

SUPER CONGRUENCES INVOLVING MULTIPLE HARMONIC SUMS AND BERNOULLI NUMBERS

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ABSTRACT. Let m, r and n be positive integers. We denote by $\mathbf{k} \vdash n$ any tuple of odd positive integers $\mathbf{k} = (k_1, \dots, k_t)$ such that $k_1 + \dots + k_t = n$ and $k_j \geq 3$ for all j . In this paper we prove that for every sufficiently large prime p

$$\sum_{\substack{l_1+l_2+\dots+l_n=mp^r \\ p \nmid l_1 l_2 \dots l_n}} \frac{1}{l_1 l_2 \dots l_n} \equiv p^{r-1} \sum_{\mathbf{k} \vdash n} C_{m, \mathbf{k}} B_{p-\mathbf{k}} \pmod{p^r}$$

where $B_{p-\mathbf{k}} = B_{p-k_1} B_{p-k_2} \dots B_{p-k_t}$ are products of Bernoulli numbers and the coefficients $C_{m, \mathbf{k}}$ are polynomials of m independent of p and r . This generalizes previous results by many different authors and confirms a conjecture by the authors and their collaborators.

1. INTRODUCTION

The Bernoulli numbers, defined by the generating series

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!},$$

have a long and intriguing history in the study of number theory, with over 3000 related papers written so far according to the online Bernoulli Number archive maintained by Dilcher and Slavutskii [3]. In modern mathematics, the Bernoulli numbers have appeared in the Euler-Maclaurin summation formula, Herbrand's Theorem concerning the class group of cyclotomic number fields, and even the Kervaire–Milnor formula in topology.

Well-documented history indicates that Jakob Bernoulli, after whom the Bernoulli numbers are named, was very proud of his discovery that sums of powers of positive integers can be quickly calculated by using these numbers. This result was independently discovered by Seki around the same time [1]. By using Fermat's Little Theorem, the formula further leads to many congruences and even super congruences involving multiple harmonic sums, which were first studied independently by the second author in [17, 18] and Hoffman in [5]. See [20, Ch. 8] for more details.

Let \mathbb{N} and \mathbb{N}_0 be the set of positive integers and nonnegative integers, respectively. For any $n, d \in \mathbb{N}$ and $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$ we define the *multiple harmonic*

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sums (MHSs) and their p -restricted version for primes p by

$$\mathcal{H}_n(\mathbf{s}) := \sum_{0 < k_1 < \dots < k_d < n} \frac{1}{k_1^{s_1} \dots k_d^{s_d}}, \quad \mathcal{H}_n^{(p)}(\mathbf{s}) := \sum_{\substack{0 < k_1 < \dots < k_d < n \\ p \nmid k_1, \dots, p \nmid k_d}} \frac{1}{k_1^{s_1} \dots k_d^{s_d}}.$$

Here, d is called the depth and $|\mathbf{s}| := s_1 + \dots + s_d$ the weight of the MHS. For example, $\mathcal{H}_{n+1}(1)$ is often called the n th harmonic number. In general, as $n \rightarrow \infty$ we see that $\mathcal{H}_n(\mathbf{s}) \rightarrow \zeta(\mathbf{s})$ which are the multiple zeta values (MZVs).

More than a decade ago, the second author discovered the curious congruence (see [16])

$$\sum_{\substack{i+j+k=p \\ i,j,k > 0}} \frac{1}{ijk} \equiv -2B_{p-3} \pmod{p} \quad (1)$$

for all primes $p \geq 3$. Since then several different types of generalizations have been found, see, for e.g. [8, 10, 12, 13, 15, 19, 21]. In this paper, we will concentrate on congruences of the following type of sums. Let \mathcal{P}_p be the set of positive integers not divisible by p . For all positive integers r and m such that $p \nmid m$, define

$$R_n^{(m)}(p^r) := \sum_{\substack{l_1+l_2+\dots+l_n=mp^r \\ l_1, \dots, l_n \in \mathcal{P}_p}} \frac{1}{l_1 l_2 \dots l_n},$$

$$S_n^{(m)}(p^r) := \sum_{\substack{l_1+l_2+\dots+l_n=mp^r \\ p^r > l_1, \dots, l_n \in \mathcal{P}_p}} \frac{1}{l_1 l_2 \dots l_n}.$$

To put these sums into proper framework, we now recall briefly the definition of the finite MZVs. Let \mathfrak{P} be the set of rational primes. To study the congruences of MHSs, Kaneko and Zagier [6] consider the following ring structure¹ first used by Kontsevich [7]:

$$\mathcal{A}_\ell := \prod_{p \in \mathfrak{P}} (\mathbb{Z}/p^\ell \mathbb{Z}) \bigg/ \bigoplus_{p \in \mathfrak{P}} (\mathbb{Z}/p^\ell \mathbb{Z}).$$

Two elements in \mathcal{A}_ℓ are the same if they differ at only finitely many components. For simplicity, we often write p^r for the element $(p^r)_{p \in \mathfrak{P}} \in \mathcal{A}_\ell$ for all positive integers $r < \ell$. For other properties and facts of \mathcal{A}_ℓ we refer the interested reader to [20, Ch. 8].

One now defines the finite MZVs as the following elements in \mathcal{A}_ℓ :

$$\zeta_{\mathcal{A}_\ell}(\mathbf{s}) := \left(\mathcal{H}_p(\mathbf{s}) \pmod{p^\ell} \right)_{p \in \mathfrak{P}}.$$

It turns out that Bernoulli numbers often play important roles in the study of finite MZVs, as witnessed by the following result (see [21, p. 1332]):

$$\zeta_{\mathcal{A}_3}(1_n) = (-1)^{n-1} \frac{(n+1)}{2} \beta_{n+2} \cdot p^2 \quad \text{if } 2 \nmid n;$$

$$\zeta_{\mathcal{A}_2}(1_n) = (-1)^n \beta_{n+1} \cdot p \quad \text{if } 2 \mid n,$$

¹More precisely, they consider only the case when $\ell = 1$.

where 1_n is the string $(1, \dots, 1)$ with 1 repeating n times, and $\beta_k := (-B_{p-k}/k \pmod{p})_{p>k} \in \mathcal{A}_1$ is the so-called \mathcal{A} -Bernoulli number, which is the finite analog of $\zeta(k)$. Note that $\beta_k = 0$ for all even positive integers k while it is still a mystery whether $\beta_k \neq 0$ for all odd integers $k > 2$.

In [8], the second author and his collaborators made the following conjecture.

Conjecture 1.1. *For any $m, n \in \mathbb{N}$, both $R_n^{(m,1)}$ and $S_n^{(m,1)}$ are elements in the sub-algebra of \mathcal{A}_1 generated by the \mathcal{A} -Bernoulli numbers.*

In this paper, we will prove this conjecture. More precisely, we have

Main Theorem. *Let m, r and n be positive integers. We denote by $\mathbf{k} \vdash n$ any tuple of odd positive integers $\mathbf{k} = (k_1, \dots, k_t)$ such that $k_1 + \dots + k_t = n$ and $k_j \geq 3$ for all j . Then for every sufficiently large prime p*

$$R_n^{(m)}(p^r) \equiv \sum_{\substack{l_1+l_2+\dots+l_n=mp^r \\ p \nmid l_1 l_2 \dots l_n}} \frac{1}{l_1 l_2 \dots l_n} \equiv p^{r-1} \sum_{\mathbf{k} \vdash n} C_{m,\mathbf{k}} B_{p-\mathbf{k}} \pmod{p^r}, \quad (2)$$

$$S_n^{(m)}(p^r) \equiv \sum_{\substack{l_1+l_2+\dots+l_n=mp^r \\ p \nmid l_1 l_2 \dots l_n}} \frac{1}{l_1 l_2 \dots l_n} \equiv p^{r-1} \sum_{\mathbf{k} \vdash n} C'_{m,\mathbf{k}} B_{p-\mathbf{k}} \pmod{p^r}, \quad (3)$$

where $B_{p-\mathbf{k}} = B_{p-k_1} B_{p-k_2} \dots B_{p-k_t}$ are products of Bernoulli numbers and the coefficients $C_{m,\mathbf{k}}$ and $C'_{m,\mathbf{k}}$ are polynomials of m independent of p and r .

The coefficients $C_{m,\mathbf{k}}$ and $C'_{m,\mathbf{k}}$ are intimately related, see Conjecture 4.4.

As a side remark, in our numerical computation, it is crucial to use some generating functions of $R_n^{(m)}$ and $S_n^{(m)}$, which are certain products of a finite variation of the p -restricted classical polylogarithm function. Unfortunately, it seems difficult to use these generating functions to obtain our main result of this paper.

2. PRELIMINARY LEMMAS

In this section, we collect some useful results to be applied in the rest of the paper.

Lemma 2.1. (cf. [8, Lemma 3.4]) *Let p be a prime, κ, s_1, \dots, s_d be positive integers, and α a non-negative integer. We define the un-ordered sum*

$$U_{\alpha;\kappa}^{(p)}(s_1, \dots, s_d) := \sum_{\substack{\alpha p < l_1, \dots, l_d < (\alpha+\kappa)p \\ l_1, \dots, l_d \in \mathcal{P}_p, l_i \neq l_j \forall i \neq j}} \frac{1}{l_1^{s_1} \dots l_d^{s_d}}.$$

If the weight $w = s_1 + \dots + s_d \leq p-3$ then we have

$$U_{\alpha;\kappa}^{(p)}(s_1, \dots, s_d) \equiv (-1)^{d-1} (d-1)! \frac{\kappa w}{w+1} B_{p-w-1} \cdot p \pmod{p^2}.$$

Lemma 2.2. *Suppose $a, k, m, n, r \in \mathbb{N}$ and p is a prime. Set*

$$\gamma_n^{(m)}(a) := (-1)^{m+a} \binom{n-2}{m-1} \frac{(a-1)!(n-1-a)!}{(n-1)!}.$$

If $k < n < p-1$ then we have

- (i) $S_n^{(k)}(p^r) \equiv (-1)^n S_n^{(n-k)}(p^r) \pmod{p^r};$
- (ii) $S_n^{(m)}(p^{r+1}) \equiv p \sum_{a=1}^{n-1} \gamma_n^{(m)}(a) S_n^{(a)}(p^r) \pmod{p^{r+1}};$
- (iii) $S_n^{(m)}(p^{r+1}) \equiv (-1)^{m-1} \binom{n-2}{m-1} S_n^{(1)}(p^2) p^{r-1} \pmod{p^{r+1}}.$

Proof. (i) and (ii) follow from [8, Lemma 2.3] while (iii) from [2, Lemma 2.2]. \square

Lemma 2.3. ([2, Proposition 2.3]) *Let $m, n, r \in \mathbb{N}$. For all $r \geq 2$, we have*

$$R_n^{(m,r)} = m \cdot S_n^{(1,2)} p^{r-2} \in \mathcal{A}_r.$$

Lemma 2.4. *Suppose $m, n, r \in \mathbb{N}$. Then we have*

$$S_n^{(m,r)} = \sum_{k=0}^{m-1} (-1)^k \binom{n}{k} R_n^{(m-k,r)} \in \mathcal{A}_r. \quad (4)$$

Proof. Equation (4) can be proved using the Inclusion-Exclusion Principle similar to the proof of [14, Lemma 1]. Indeed, for all primes p

$$\begin{aligned}
S_n^{(m)}(p^r) &= \sum_{\substack{l_1+\dots+l_n=mp^r \\ l_1, \dots, l_n \in \mathcal{P}_p}} \frac{1}{l_1 \cdots l_n} + \sum_{k=1}^{m-1} (-1)^k \sum_{\substack{1 \leq a_1 < \dots < a_k \leq n \\ l_1+\dots+l_n=mp^r \\ l_1, \dots, l_n \in \mathcal{P}_p \\ l_{a_1} > p^r, \dots, l_{a_k} > p^r}} \frac{1}{l_1 \cdots l_n} \\
&= \sum_{k=0}^{m-1} (-1)^k \binom{n}{k} \sum_{\substack{l_1+\dots+l_n=(m-k)p^r \\ l_1, \dots, l_n \in \mathcal{P}_p}} \frac{1}{(l_1 + p^r) \cdots (l_k + p^r) l_{k+1} \cdots l_n} \\
&\equiv \sum_{k=0}^{m-1} (-1)^k \binom{n}{k} \sum_{\substack{l_1+\dots+l_n=(m-k)p^r \\ l_1, \dots, l_n \in \mathcal{P}_p}} \frac{1}{l_1 \cdots l_n} \pmod{p^r} \\
&\equiv \sum_{k=0}^{m-1} (-1)^k \binom{n}{k} R_n^{(m-k)}(p^r) \pmod{p^r},
\end{aligned}$$

as desired. \square

We see immediately from Lemmas 2.3 and 2.4 that the proof of the Main Theorem is reduced to its special case of $S_n^{(1,2)}$. The idea is to compute $R_n^{(m,1)}$ first, which leads to $S_n^{(m,1)}$ by the Lemma 2.4. Then $S_n^{(1,2)}$ can be determined using $S_n^{(m,1)}$ by Lemma 2.2 (ii).

For the convenience of numerical computation, we list some of the relevant known results.

Lemma 2.5. ([21, Main Theorem]) *Let $n > 1$ be positive integer. Then we have*

$$S_n^{(1,1)} = R_n^{(1,1)} = \begin{cases} n! \beta_n & \text{if } 2 \nmid n; \\ 0 & \text{if } 2 \mid n. \end{cases}$$

Lemma 2.6. *Let $n > 1$ be positive integer. Then we have*

$$R_n^{(2,1)} = \begin{cases} \frac{(n+1)!}{2} \beta_n & \text{if } 2 \nmid n; \\ \frac{n!}{2} \sum_{a+b=n} \beta_a \beta_b & \text{if } 2 \mid n, \end{cases}$$

and

$$S_n^{(2,1)} = \begin{cases} -\frac{n-1}{2} n! \beta_n & \text{if } 2 \nmid n; \\ \frac{n!}{2} \sum_{a+b=n} \beta_a \beta_b & \text{if } 2 \mid n. \end{cases}$$

Proof. The odd cases follow from [8, Lemma 3.5 and Cor 3.6] respectively. The even cases are proved in [14, Theorem 1 and Corollary 1]. \square

Lemma 2.7. *Let $n > 1$ be positive integer. Then we have*

$$R_n^{(3,1)} = \begin{cases} \binom{n+2}{3} \cdot (n-1)! \beta_n + \frac{n!}{6} \sum_{a+b+c=n} \beta_a \beta_b \beta_c & \text{if } 2 \nmid n; \\ \frac{n!(n+2)}{4} \sum_{a+b=n} \beta_a \beta_b & \text{if } 2 \mid n, \end{cases}$$

and

$$S_n^{(3,1)} = \begin{cases} \binom{n}{3} \cdot (n-1)! \beta_n + \frac{n!}{6} \sum_{a+b+c=n} \beta_a \beta_b \beta_c & \text{if } 2 \nmid n; \\ -\frac{n!(n-2)}{4} \sum_{a+b=n} \beta_a \beta_b & \text{if } 2 \mid n. \end{cases}$$

Proof. The odd cases follow from [8, Lemma 3.7 and Corollary 3.7] respectively. The even cases are essentially proved in [14, Theorem 2 and Corollary 2]. We only need to observe that if n is even then by exchanging the indices a and b in half of the sums, we get

$$\begin{aligned} R_n^{(3)}(p) &\equiv \frac{n!}{6} \sum_{a+b=n} (2n-a+3) \frac{B_{p-a} B_{p-b}}{ab} \pmod{p} \\ &\equiv \frac{n!}{12} \sum_{a+b=n} (4n-a-b+6) \frac{B_{p-a} B_{p-b}}{ab} \pmod{p} \\ &\equiv \frac{n!(n+2)}{4} \sum_{a+b=n} \frac{B_{p-a} B_{p-b}}{ab} \pmod{p} \end{aligned}$$

since $a+b=n$. Similarly

$$\begin{aligned} S_n^{(3)}(p) &\equiv -\frac{n!}{6} \sum_{a+b=n} (n+a-3) \frac{B_{p-a} B_{p-b}}{ab} \pmod{p} \\ &\equiv -\frac{n!}{12} \sum_{a+b=n} (2n+a+b-6) \frac{B_{p-a} B_{p-b}}{ab} \pmod{p} \end{aligned}$$

$$\equiv -\frac{n!(n-2)}{4} \sum_{a+b \vdash n} \frac{B_{p-a} B_{p-b}}{ab} \pmod{p}$$

as desired. \square

3. SUMS RELATED TO MULTIPLE HARMONIC SUMS

We are now ready to consider the sums $R_n^{(m,r)}$. The key step is to compute $R_n^{(m,1)}$ for $m \leq n/2$, which we now transform using MHSs. By the definition, for all primes p , we have

$$\begin{aligned} R_n^{(m)}(p) &= \frac{1}{mp} \sum_{\substack{l_1+l_2+\dots+l_n=mp \\ l_1, \dots, l_n \in \mathcal{P}_p}} \frac{l_1+l_2+\dots+l_n}{l_1 l_2 \dots l_n} \\ &= \frac{n}{mp} \sum_{\substack{u_{n-1}=l_1+l_2+\dots+l_{n-1} < mp \\ l_1, \dots, l_{n-1}, u_{n-1} \in \mathcal{P}_p}} \frac{1}{l_1 l_2 \dots l_{n-1}} \quad (\text{by symmetry of } l_1, \dots, l_n) \\ &= \frac{n}{mp} \sum_{\substack{u_{n-1}=l_1+l_2+\dots+l_{n-1} < mp \\ l_1, \dots, l_{n-1}, u_{n-1} \in \mathcal{P}_p}} \frac{l_1+l_2+\dots+l_{n-1}}{l_1 l_2 \dots l_{n-1} u_{n-1}} \\ &= \frac{n(n-1)}{mp} \sum_{\substack{u_{n-2}=l_1+l_2+\dots+l_{n-2} < u_{n-1} < mp \\ l_1, \dots, l_{n-2} \in \mathcal{P}_p \\ u_{n-1}-u_{n-2}, u_{n-1} \in \mathcal{P}_p}} \frac{1}{l_1 l_2 \dots l_{n-2} u_{n-1}}. \end{aligned}$$

Continuing this process by using the substitution $u_j = l_1 + l_2 + \dots + l_j$ for each $j = n-3, \dots, 2, 1$, we arrive at

$$R_n^{(m)}(p) = \frac{n!}{mp} \sum_{\substack{0 < u_1 < \dots < u_{n-1} < mp \\ u_1, u_2-u_1, \dots, u_{n-1}-u_{n-2}, u_{n-1} \in \mathcal{P}_p}} \frac{1}{u_1 u_2 \dots u_{n-1}}.$$

Observe that the indices u_j ($j = 2, \dots, n-2$) are allowed to be multiples of p . Thus we set

$$T_{n,\ell}^{(m)}(p) := \sum_{\substack{2 \leq a_1 < \dots < a_{\ell-1} \leq n-2 \\ 1 \leq k_1 < \dots < k_{\ell-1} < m}} \sum_{\substack{0 < u_1 < \dots < u_{n-1} < mp \\ u_{a_1} = k_1 p, \dots, u_{a_{\ell-1}} = k_{\ell-1} p, \\ u_j \in \mathcal{P}_p \quad \forall j \neq a_1, \dots, a_{\ell-1} \\ u_2 - u_1, \dots, u_{n-1} - u_{n-2} \in \mathcal{P}_p}} \frac{1}{u_1 \dots u_{n-1}}.$$

In this sum, the indices u_1, \dots, u_{n-1} are divided into ℓ -parts by p -multiples so that the indices inside each part (excluding the boundaries) are all prime to p . Hence we can rewrite

$$R_n^{(m)}(p) = \frac{n!}{mp} \sum_{1 \leq \ell < n/2} T_{n,\ell}^{(m)}(p). \quad (5)$$

So we are naturally led to the study of the following sums. Let $\alpha \in \mathbb{N}_0$, $\kappa, n \in \mathbb{N}$ and p be a prime. Suppose $n > 1$. Define

$$\Xi_{\alpha;\kappa}^{(p)}(n) := \sum_{\substack{\alpha p < u_1 < \dots < u_{n-1} < (\alpha + \kappa)p \\ u_1, u_2, \dots, u_{n-1} \in \mathcal{P}_p \\ u_2 - u_1, \dots, u_{n-1} - u_{n-2} \in \mathcal{P}_p}} \frac{1}{u_1 \dots u_{n-1}}.$$

For convenience, in the above sum if the difference between two adjacent indices is a multiple of p (which is of course not allowed in the definition of Ξ) we then say there is a p -gap between this pair of indices. Let $\alpha \in \mathbb{N}_0$ and $\kappa \in \mathbb{N}$. For all $1 \leq g \leq \min\{\kappa - 1, n - 2\}$, define the sum in which at least g p -gaps appear by

$$P_{\alpha;\kappa}^{g;p}(n) := \sum_{1 < b_1 < \dots < b_g < n} \sum_{\substack{\alpha p < u_1 < \dots < u_{n-1} < (\alpha + \kappa)p \\ u_1, u_2, \dots, u_{n-1} \in \mathcal{P}_p \\ p | (u_{b_1} - u_{b_1-1}), \dots, p | (u_{b_g} - u_{b_g-1})}} \frac{1}{u_1 \dots u_{n-1}}. \quad (6)$$

The following technical result is crucial in the proof of our Main Theorem.

Proposition 3.1. *Let $\alpha \in \mathbb{N}_0$, $\kappa, n \in \mathbb{N}$. Then, for all $1 \leq g \leq \min\{\kappa - 1, n - 2\}$, we have*

$$P_{\alpha;\kappa}^{g;p}(n) \equiv P_{0;\kappa}^{g;p}(n) \equiv -(-1)^g \binom{\kappa}{g+1} \binom{n-1}{g} \frac{B_{p-n}}{n} p \pmod{p^2}. \quad (7)$$

We postpone the proof of this proposition to the next section due to its length. A direct consequence is the following corollary.

Corollary 3.2. *Let $\alpha, \kappa, n \in \mathbb{N}$. Then for all primes $p > n + 1$, we have*

$$\Xi_{\alpha;\kappa}^{(p)}(n) \equiv \left[\binom{\kappa}{n} - \binom{\kappa + n - 1}{n} \right] \frac{B_{p-n}}{n} p \pmod{p^2}.$$

Proof. Set $\delta_{n>j} = 1$ if $n > j$ and $\delta_{n>j} = 0$ if $n \leq j$. By the Inclusion-Exclusion Principle it is clear that

$$\begin{aligned} \Xi_{\alpha;\kappa}^{(p)}(n) &= \frac{U_{\alpha;\kappa}^{(p)}(1_{n-1})}{(n-1)!} + \sum_{g=1}^{\kappa-1} (-1)^g \delta_{n>g+1} P_{\alpha;\kappa}^{g;p}(n) \\ &\equiv - \sum_{h=1}^{\kappa} \delta_{n>h} \binom{\kappa}{h} \binom{n-1}{h-1} \frac{B_{p-n}}{n} p \pmod{p^2} \\ &\equiv \left[\binom{\kappa}{n} - \sum_{h=1}^{\kappa} \binom{\kappa}{h} \binom{n-1}{h-1} \right] \frac{B_{p-n}}{n} p \pmod{p^2} \end{aligned}$$

by (7) and the congruence (see Lemma 2.1)

$$\frac{U_{0;\kappa}^{(p)}(1_{n-1})}{(n-1)!} \equiv - \frac{\kappa B_{p-n}}{n} p \pmod{p^2}.$$

So the proposition follows immediately from the well-known binomial identity

$$\sum_{h=1}^{\kappa} \binom{\kappa}{h} \binom{n-1}{h-1} = \sum_{h=1}^{\kappa} \binom{\kappa}{h} \binom{n-1}{n-h} = \binom{\kappa + n - 1}{n}.$$

□

By Corollary 3.2, it is easy to see that for any fixed $\ell < n/2$

$$\begin{aligned}
T_{n,\ell}^{(m)}(p) &\equiv \sum_{\substack{1 \leq a_1 < \dots < a_{\ell-1} < n \\ 1 \leq k_1 < \dots < k_{\ell-1} < m}} \left(\prod_{j=1}^{\ell-1} \frac{1}{k_j p} \right) \left(\prod_{j=1}^{\ell} \Xi_{k_{j-1}; k_j - k_{j-1}}^{(p)} (a_j - a_{j-1}) \right) \\
&\equiv p \sum_{\substack{k_1 + \dots + k_{\ell} = m \\ k_1, \dots, k_{\ell} \geq 1 \\ a_1 + \dots + a_{\ell} \vdash n}} \prod_{j=1}^{\ell-1} \frac{1}{k_1 + \dots + k_j} \prod_{j=1}^{\ell} \left[\binom{k_j}{a_j} - \binom{k_j + a_j - 1}{a_j} \right] \frac{B_{p-a_j}}{a_j} \quad (8)
\end{aligned}$$

modulo p^2 , where we have set $k_0 = a_0 = 0$ and $k_{\ell} = m, a_{\ell} = n$. In the last step above, we have used substitutions $k_j \rightarrow k_1 + \dots + k_j$ and $a_j \rightarrow a_1 + \dots + a_j$ for all $j \leq \ell - 1$. In view of (5) and Lemma 2.4, we easily obtain the following result which confirms Conjecture 1.1.

Theorem 3.3. *For all positive integer m and n , we have*

$$R_n^{(m,1)} = \frac{n!}{m} \sum_{\substack{1 \leq \ell \leq \lfloor n/3 \rfloor \\ k_1 + \dots + k_{\ell} = m, k_j \geq 1 \forall j \\ a_1 + \dots + a_{\ell} \vdash n}} \prod_{j=1}^{\ell-1} \frac{1}{k_1 + \dots + k_j} \prod_{j=1}^{\ell} \left[\binom{k_j + a_j - 1}{a_j} - \binom{k_j}{a_j} \right] \beta_{a_j}.$$

4. SOME NUMERICAL EXAMPLES

Using the formula of Theorem 3.3, we obtain the following results which extend those in Lemmas 2.5, 2.6 and 2.7. To guarantee accuracy, we have checked these congruences for $m, n \leq 20$ and primes $p < 100$ using Maple.

Corollary 4.1. *For any $\kappa, m, n \in \mathbb{N}$, we have*

$$R_3^{(m,1)} = 3!m\beta_3, \quad R_5^{(m,1)} = \frac{5!}{3!}m(m^2 + 5)\beta_5, \quad R_7^{(m,1)} = \frac{7!}{5!}m(m^4 + 35m^2 + 84)\beta_7. \quad (9)$$

If $n \geq 9$ is odd then

$$\begin{aligned}
R_n^{(4,1)} &= (n-1)! \binom{n+3}{4} \beta_n + \frac{n!(n+3)}{2 \cdot 3!} \sum_{a+b+c \vdash n} \beta_a \beta_b \beta_c, \\
R_n^{(5,1)} &= (n-1)! \binom{n+4}{5} \beta_n + \frac{n!}{5!} \sum_{a_1 + \dots + a_5 \vdash n} \beta_{a_1} \dots \beta_{a_5} \\
&\quad + \frac{n!}{4!} \sum_{a+b+c \vdash n} \left(\frac{n^2}{2} + 4n + 7 + \frac{a^2}{2} - 4 \binom{3}{a} \right) \beta_a \beta_b \beta_c, \\
R_n^{(6,1)} &= (n-1)! \binom{n+5}{6} \beta_n + \frac{n!(n+5)}{2 \cdot 5!} \sum_{a_1 + \dots + a_5 \vdash n} \beta_{a_1} \dots \beta_{a_5} \\
&\quad + \frac{n!}{4 \cdot 4!} \sum_{a+b+c \vdash n} \left(\frac{n^3}{3} + a^3 + 2a^2b + 5n^2 + 5a^2 + \frac{68n}{3} + 30 - 8 \binom{3}{a} (n+5) \right) \beta_a \beta_b \beta_c.
\end{aligned}$$

If $n \geq 2$ is even then

$$\begin{aligned}
R_n^{(4,1)} &= \frac{n!}{4!} \left(\sum_{a+b \vdash n} \left(\frac{3}{2}n^2 + 9n + 11 + a^2 - 8 \binom{3}{a} \right) \beta_a \beta_b + \sum_{a+b+c+d \vdash n} \beta_a \beta_b \beta_c \beta_d \right), \\
R_n^{(5,1)} &= \frac{n!}{3 \cdot 4!} \sum_{a+b \vdash n} \left(n^3 + 9n^2 + \frac{63}{2}n + 30 + a^3 + 6a^2 - 12(n+4) \binom{3}{a} \right) \beta_a \beta_b \\
&\quad + \frac{n!(n+4)}{2 \cdot 4!} \sum_{a+b+c+d \vdash n} \beta_a \beta_b \beta_c \beta_d, \\
R_n^{(6,1)} &= \frac{n!}{6!} \sum_{a+b+c+d+e+f \vdash n} \beta_a \beta_b \beta_c \beta_d \beta_e \beta_f \\
&\quad + \frac{n!}{6} \sum_{a+b \vdash n} \left[\frac{1}{3} \binom{3}{a} \binom{3}{b} - \frac{6}{5} \binom{5}{a} - \frac{n^2 + 6n - 16}{3} \binom{3}{a} + \frac{1}{5!} \left(\frac{8}{3}a^4 + \right. \right. \\
&\quad \left. \left. + 25a^3 + 85a^2 + \frac{675n}{2} + 274 + \frac{5}{6}a^3n + \frac{5}{3}n^4 + 25n^3 + \frac{255}{2}n^2 \right) \right] \beta_a \beta_b \\
&\quad + \frac{n!}{144} \sum_{a+b+c+d \vdash n} \left[a^2 + \frac{3n^2}{4} + \frac{15n}{2} + 17 - 8 \binom{3}{a} \right] \beta_a \beta_b \beta_c \beta_d.
\end{aligned}$$

Example 4.2. When $8 \leq n \leq 12$ we get, respectively,

$$\begin{aligned}
R_8^{(4,1)} &= 16 \cdot 8! \beta_3 \beta_5, & R_9^{(4,1)} &= 9! (55 \beta_9 + \beta_3^3), \\
R_{10}^{(5,1)} &= 35 \cdot 10! (2 \beta_3 \beta_7 + \beta_5^2), & R_{11}^{(5,1)} &= 11! \left(273 \beta_{11} + \frac{29}{2} \beta_3^2 \beta_5 \right), \\
R_{12}^{(6,1)} &= 12! \left(333 \beta_3 \beta_9 + 321 \beta_5 \beta_7 + \frac{3}{2} \beta_3^4 \right).
\end{aligned}$$

The first three identities were predicted by [8, Conjecture 5.1]. The last two were also discovered numerically earlier [19, Conjecture 7.2].

Corollary 4.3. Let $n \geq 2$ be a positive integer. If n is odd then

$$\begin{aligned}
S_n^{(4,1)} &= - (n-1)! \binom{n}{4} \beta_n - \frac{n!(n-3)}{12} \sum_{a+b+c \vdash n} \beta_a \beta_b \beta_c, \\
S_n^{(5,1)} &= (n-1)! \binom{n}{5} \beta_n + \frac{n!}{5!} \sum_{a_1+\dots+a_5 \vdash n} \beta_{a_1} \cdots \beta_{a_5} \\
&\quad + \frac{n!}{4!} \sum_{a+b+c \vdash n} \left(\frac{n^2}{2} - 4n + 7 + \frac{a^2}{2} - 4 \binom{3}{a} \right) \beta_a \beta_b \beta_c, \\
S_n^{(6,1)} &= - (n-1)! \binom{n}{6} \beta_n - \frac{n-5}{2 \cdot 5!} \sum_{a_1+\dots+a_5 \vdash n} \beta_{a_1} \cdots \beta_{a_5} \\
&\quad - \frac{1}{96} \sum_{a+b+c \vdash n} \left(\frac{n^3}{3} + a^3 - 2a^2b - 5n^2 + 5a^2 + \frac{68n}{3} - 30 - 8 \binom{3}{a} (n-5) \right) \beta_a \beta_b \beta_c.
\end{aligned}$$

If n is even then

$$\begin{aligned}
S_n^{(4,1)} &= \frac{n!}{4!} \sum_{a+b \vdash n} \left(\frac{3}{2}n^2 - 9n + 11 + a^2 - 8 \binom{3}{a} \right) \beta_a \beta_b + \frac{n!}{4!} \sum_{a+b+c+d \vdash n} \beta_a \beta_b \beta_c \beta_d, \\
S_n^{(5,1)} &= \frac{n!}{144} \sum_{a+b \vdash n} \left(-2n^3 + 18n^2 - 63n + 60 - 2a^3 + 12a^2 + 24(n-4) \binom{3}{a} \right) \beta_a \beta_b \\
&\quad - \frac{n!(n-4)}{48} \sum_{a+b+c+d \vdash n} \beta_a \beta_b \beta_c \beta_d, \\
S_n^{(6,1)} &= \frac{n!}{6!} \sum_{a+b+c+d+e+f \vdash n} \beta_a \beta_b \beta_c \beta_d \beta_e \beta_f \\
&\quad + \frac{n!}{6} \sum_{a+b \vdash n} \left[\frac{1}{3} \binom{3}{a} \binom{3}{b} - \frac{6}{5} \binom{5}{a} - \frac{n^2 - 9n - 16}{3} \binom{3}{a} + \frac{1}{5!} \left(\frac{8}{3}a^4 \right. \right. \\
&\quad \left. \left. - 25a^3 + 85a^2 - \frac{675n}{2} + 274 + \frac{5}{6}a^3n + \frac{5}{3}n^4 - 25n^3 + \frac{255}{2}n^2 \right) \right] \beta_a \beta_b \\
&\quad + \frac{n!}{144} \sum_{a+b+c+d \vdash n} \left[a^2 + \frac{3n^2}{4} - \frac{15n}{2} + 17 - 8 \binom{3}{a} \right] \beta_a \beta_b \beta_c \beta_d.
\end{aligned}$$

Proof. By Lemma 2.4, we see that

$$S_n^{(4,1)} = R_n^{(4,1)} - nR_n^{(3,1)} + \binom{n}{2}R_n^{(2,1)} - \binom{n}{3}R_n^{(1,1)}.$$

Thus the statements concerning $S_n^{(4,1)}$ follow from Lemma 2.5, 2.6, 2.7 and Corollary 4.1 immediately. The computation of $S_n^{(5,1)}$ and $S_n^{(6,1)}$ can be done similarly. So we leave them to the interested reader. \square

By comparing the above two corollaries, we can formulate the following conjecture.

Conjecture 4.4. For all $m, n \in \mathbb{N}$, suppose

$$R_n^{(m,1)} = n! \sum_{1 \leq l \leq n/3, 2|(n-l)} \sum_{a_1 + \dots + a_l \vdash n} C(a_1, \dots, a_l) \beta_{a_1} \dots \beta_{a_l}.$$

Then

$$S_n^{(m,1)} = n! \sum_{1 \leq l \leq n/3, 2|(n-l)} \sum_{a_1 + \dots + a_l \vdash n} C(-a_1, \dots, -a_l) \beta_{a_1} \dots \beta_{a_l}.$$

Corollary 4.5. For all $r \geq 2$, we have

$$\begin{aligned}
S_8^{(m,r)} &= (-1)^m \binom{6}{m-1} 5376 \beta_3 \beta_5 p^{r-1} && \in \mathcal{A}_r \quad \forall m \leq 7, \\
S_9^{(m,r)} &= (-1)^{m-1} \binom{7}{m-1} 36(6088 \beta_9 + 61 \beta_3^3) p^{r-1} && \in \mathcal{A}_r \quad \forall m \leq 8, \\
S_{10}^{(m,r)} &= (-1)^m \binom{8}{m-1} 223200(\beta_5^2 + 2\beta_3 \beta_7) p^{r-1} && \in \mathcal{A}_r \quad \forall m \leq 9,
\end{aligned}$$

$$S_{11}^{(m,r)} = (-1)^{m-1} \binom{9}{m-1} 174240(122\beta_{11} + 3\beta_3^2\beta_5)p^{r-1} \in \mathcal{A}_r \quad \forall m \leq 10,$$

$$S_{12}^{(m,r)} = (-1)^m \binom{10}{m-1} 47520(896\beta_3\beta_9 + 872\beta_5\beta_7 + 3\beta_3^4)p^{r-1} \in \mathcal{A}_r \quad \forall m \leq 11.$$

Proof. Let p be a prime such that $p \geq 17$. By Lemma 2.5, 2.6, 2.7, and Corollary 4.3, we have modulo p

$$\begin{aligned} S_8^{(1)}(p) &\equiv 0, \quad S_8^{(2)}(p) \equiv \frac{8!}{15} B_{p-3} B_{p-5}, \quad S_8^{(3)}(p) \equiv -3S_8^{(2)}(p), \quad S_8^{(4)}(p) \equiv 4S_8^{(2)}(p), \\ S_9^{(1)}(p) &\equiv -8!B_{p-9}, \quad S_9^{(2)}(p) \equiv 4 \cdot 8!B_{p-9}, \\ S_9^{(3)}(p) &\equiv -\frac{8!}{18} B_{p-3}^3 - \frac{28 \cdot 8!}{3} B_{p-9}, \quad S_9^{(4)}(p) \equiv \frac{8!}{6} B_{p-3}^3 + 14 \cdot 8!B_{p-9}, \\ S_{10}^{(1)}(p) &\equiv 0, \quad S_{10}^{(2)}(p) \equiv \frac{1}{2} \cdot 10! \left(\frac{B_{p-5}^2}{25} + \frac{2B_{p-3}B_{p-7}}{21} \right), \\ S_{10}^{(3)}(p) &\equiv -4S_{10}^{(2)}(p), \quad S_{10}^{(4)}(p) \equiv 8S_{10}^{(2)}(p), \quad S_{10}^{(5)}(p) \equiv -10S_{10}^{(2)}(p), \\ S_{11}^{(1)}(p) &\equiv -10!B_{p-11}, \quad S_{11}^{(2)}(p) \equiv 5 \cdot 10!B_{p-11}, \\ S_{11}^{(3)}(p) &\equiv -\frac{11!}{90} B_{p-3}^2 B_{p-5} - 15 \cdot 10!B_{p-11}, \\ S_{11}^{(4)}(p) &\equiv \frac{2 \cdot 11!}{45} B_{p-3}^2 B_{p-5} + 30 \cdot 10!B_{p-11}, \\ S_{11}^{(5)}(p) &\equiv -\frac{7 \cdot 11!}{90} B_{p-3}^2 B_{p-5} - 42 \cdot 10!B_{p-11}, \\ S_{12}^{(1)}(p) &\equiv 0, \quad S_{12}^{(2)}(p) \equiv \frac{12!}{27} B_{p-3} B_{p-9} + \frac{12!}{35} B_{p-5} B_{p-7}, \quad S_{12}^{(3)}(p) \equiv -5S_{12}^{(2)}(p), \\ S_{12}^{(4)}(p) &\equiv \frac{40 \cdot 12! B_{p-3} B_{p-9}}{81} + 13 \cdot 12! \frac{B_{p-5} B_{p-7}}{35} + \frac{12!}{24} \frac{B_{p-3}^4}{3^4}, \\ S_{12}^{(5)}(p) &\equiv -\frac{70 \cdot 12! B_{p-3} B_{p-9}}{81} - 22 \cdot 12! \frac{B_{p-5} B_{p-7}}{35} - \frac{12!}{6} \frac{B_{p-3}^4}{3^4}, \\ S_{12}^{(6)}(p) &\equiv \frac{28 \cdot 12! B_{p-3} B_{p-9}}{27} + 26 \cdot 12! \frac{B_{p-5} B_{p-7}}{35} + \frac{12!}{4} \frac{B_{p-3}^4}{3^4}. \end{aligned}$$

Taking $n = 8$ and $r = 1$ in Lemma 2.2 (i) and (ii), we get

$$\begin{aligned} S_8^{(1)}(p^2) &\equiv \frac{2p}{7} S_8^{(1)}(p) - \frac{p}{21} S_8^{(2)}(p) + \frac{2p}{105} S_8^{(3)}(p) - \frac{p}{140} S_8^{(4)}(p) \\ &\equiv -\frac{1792}{5} p B_{p-3} B_{p-5} \pmod{p^2}. \end{aligned}$$

Similarly, taking $9 \leq n \leq 12$ and $r = 1$ in Lemma 2.2 (i) and (ii), we see that

$$\begin{aligned} S_9^{(1)}(p^2) &\equiv \frac{p}{4} S_9^{(1)}(p) - \frac{p}{28} S_9^{(2)}(p) + \frac{p}{84} S_9^{(3)}(p) - \frac{p}{140} S_9^{(4)}(p) \\ &\equiv -288 \left(\frac{761 B_{p-9}}{9} + \frac{7 B_{p-3}^3}{3^3} \right) p \pmod{p^2}, \end{aligned}$$

$$\begin{aligned}
S_{10}^{(1)}(p^2) &\equiv \frac{2p}{9}S_{10}^{(1)}(p) - \frac{p}{36}S_{10}^{(2)}(p) + \frac{p}{126}S_{10}^{(3)}(p) - \frac{p}{252}S_{10}^{(4)}(p) + \frac{p}{630}S_{10}^{(5)}(p) \\
&\equiv -194400 \left(\frac{B_{p-5}^2}{25} + \frac{2B_{p-3}B_{p-7}}{21} \right) p \pmod{p^2}, \\
S_{11}^{(1)}(p^2) &\equiv \frac{p}{5}S_{11}^{(1)}(p) - \frac{p}{45}S_{11}^{(2)}(p) + \frac{p}{180}S_{11}^{(3)}(p) - \frac{p}{420}S_{11}^{(4)}(p) + \frac{p}{630}S_{11}^{(5)}(p) \\
&\equiv -174240 \left(\frac{122B_{p-11}}{11} + \frac{3B_{p-3}^2B_{p-5}}{45} \right) p \pmod{p^2}, \\
S_{12}^{(1)}(p^2) &\equiv \frac{2p}{11}S_{12}^{(1)}(p) - \frac{p}{55}S_{12}^{(2)}(p) + \frac{2p}{495}S_{12}^{(3)}(p) - \frac{p}{660}S_{12}^{(4)}(p) \\
&\quad + \frac{p}{1155}S_{12}^{(5)}(p) - \frac{p}{2772}S_{12}^{(6)}(p) \\
&\equiv -47520 \left(\frac{896B_{p-3}B_{p-9}}{27} + \frac{872B_{p-5}B_{p-7}}{35} + \frac{3B_{p-3}^4}{3^4} \right) p \pmod{p^2}.
\end{aligned}$$

Now the corollary follows quickly from Lemma 2.2 (iii). \square

Corollary 4.6. *For all positive integers $m \geq 1$, we have*

$$\begin{aligned}
R_8^{(m,1)} &= 336m(m^2 + 16)(m^2 - 1)\beta_3\beta_5, \\
R_9^{(m,1)} &= 12 \cdot 7! \binom{m+2}{5} \beta_3^3 + 72m(m^6 + 126m^4 + 1869m^2 + 3044)\beta_9, \\
R_{10}^{(m,1)} &= 360m(m^2 - 1)(m^4 + 71m^2 + 540)(2\beta_3\beta_7 + \beta_5^2), \\
R_{11}^{(m,1)} &= 660 \cdot 5! \binom{m+2}{5} (m^2 + 33)\beta_3^2\beta_5 \\
&\quad + 110m(m^8 + 330m^6 + 16401m^4 + 152900m^2 + 193248)\beta_{11}, \\
R_{12}^{(m,1)} &= 55 \cdot 9! \binom{m+3}{7} \beta_3^4 \\
&\quad + 11 \cdot 6! \binom{m+1}{3} (m^6 + 211m^4 + 6196m^2 + 32256)\beta_3\beta_9 \\
&\quad + 11 \cdot 6! \binom{m+1}{3} (m^6 + 187m^4 + 6508m^2 + 31392)\beta_5\beta_7.
\end{aligned}$$

Proof. Let p be a prime such that $p \geq 11$. By Lemma 2.3 and Corollary 4.5, we have

$$R_8^{(m)}(p) \equiv \sum_{a=1}^7 \binom{m+7-a}{7} S_8^{(a)}(p) \equiv \frac{112}{5}m(m^2 + 16)(m^2 - 1)B_{p-3}B_{p-5} \pmod{p}.$$

Similarly,

$$\begin{aligned}
R_9^{(m)}(p) &\equiv -\frac{8!}{18} \binom{m+2}{5} B_{p-3}^3 - 8m(m^6 + 126m^4 + 1869m^2 + 3044)B_{p-9} \pmod{p}, \\
R_{10}^{(m)}(p) &\equiv \frac{10!}{10080}m(m^2 - 1)(m^4 + 71m^2 + 540) \left(\frac{2B_{p-3}B_{p-7}}{21} + \frac{B_{p-5}^2}{25} \right) \pmod{p},
\end{aligned}$$

$$\begin{aligned}
 R_{11}^{(m)}(p) &\equiv -88 \cdot 5! \binom{m+2}{5} (m^2 + 33) B_{p-3}^2 B_{p-5} \\
 &\quad - 10m(m^8 + 330m^6 + 16401m^4 + 152900m^2 + 193248) B_{p-11} \pmod{p}, \\
 R_{12}^{(m)}(p) &\equiv \frac{55 \cdot 8!}{9} \binom{m+3}{7} B_{p-3}^4 \\
 &\quad + \frac{22 \cdot 5!}{9} \binom{m+1}{3} (m^6 + 211m^4 + 6196m^2 + 32256) B_{p-3} B_{p-9} \\
 &\quad + \frac{66 \cdot 4!}{7} \binom{m+1}{3} (m^6 + 187m^4 + 6508m^2 + 31392) B_{p-5} B_{p-7} \pmod{p}.
 \end{aligned}$$

The corollary now quickly follows from the definition of β_k . \square

5. PROOF OF PROPOSITION 3.1 AND MAIN THEOREM

We first deal with the case $\alpha = 0$ and rewrite it as a difference of two sums each of which can be computed more easily.

Let $n, \kappa, g \in \mathbb{N}$ such that $1 \leq g \leq \min\{\kappa - 1, n - 2\}$. Set $d = n - g - 1$. For any prime p , we define

$$V_{\kappa}^{g;p}(n) := \sum_{\substack{0 < a_1 < \dots < a_g < \kappa \\ 0 < b_1 \leq \dots \leq b_g \leq d}} \sum_{\substack{0 < u_1 < \dots < u_d < (\kappa - a_g)p \\ u_1, u_2, \dots, u_d \in \mathcal{P}_p}} \frac{1}{u_1 \dots u_d u_{b_1} \dots u_{b_g}},$$

and

$$\begin{aligned}
 M_{\kappa}^{g;p}(n) &:= \sum_{\substack{0 < a_1 < \dots < a_g < \kappa \\ 0 < b_1 \leq \dots \leq b_g \leq d}} \sum_{\substack{0 < u_1 < \dots < u_d < (\kappa - a_g)p \\ u_1, u_2, \dots, u_d \in \mathcal{P}_p}} \frac{1}{u_1 \dots u_d u_{b_1} \dots u_{b_g}} \left(\frac{a_1}{u_{b_1}} + \frac{a_1}{u_{b_1+1}} + \right. \\
 &\quad \left. \dots + \frac{a_1}{u_{b_2}} + \frac{a_2}{u_{b_2}} + \dots + \frac{a_2}{u_{b_3}} + \frac{a_3}{u_{b_3}} + \dots + \frac{a_{g-1}}{u_{b_g}} + \frac{a_g}{u_{b_g}} + \dots + \frac{a_g}{u_d} \right).
 \end{aligned}$$

Lemma 5.1. *We have*

$$V_{\kappa}^{g;p}(n) \equiv (-1)^{g+1} \binom{\kappa}{g+1} \binom{n-1}{g} \frac{B_{p-n}}{n} p \pmod{p^2}.$$

Proof. Let $d = n - g - 1$ and $m \in \mathbb{N}$. For each $0 < b_1 \leq \dots \leq b_g \leq d$, we write $\mathbf{b} = (b_1, \dots, b_g)$ and define

$$K_{d,m}^{(p)}(\mathbf{b}) := \sum_{\substack{0 < u_1 < \dots < u_d < m \\ u_1, u_2, \dots, u_d \in \mathcal{P}_p}} \frac{1}{u_1 \dots u_d u_{b_1} \dots u_{b_g}}.$$

Let $[d]^g$ be the set of g -tuples of integers in $\{1, \dots, d\}$. Let $\text{DW}(d, n-1) \subset \mathbb{N}^d$ be the set of d -tuples \mathbf{s} of positive integers with $|\mathbf{s}| = n-1$. Since every element of $[d]^g$ can be written in the form of $(1_{s_1-1}, 2_{s_2-1}, \dots, d_{s_d-1})$, we may define a map

$$\begin{aligned}
 \rho : \quad [d]^g &\longrightarrow \text{DW}(d, n) \\
 (1_{s_1-1}, 2_{s_2-1}, \dots, d_{s_d-1}) &\longmapsto (s_1, \dots, s_d).
 \end{aligned} \tag{10}$$

It's clear that ρ has an inverse so that it provides a 1-1 correspondence. Moreover,

$$K_{d;m}^{(p)}(\mathbf{b}) = \mathcal{H}_m^{(p)}(\rho(\mathbf{b})).$$

Thus, by the substitution $a_j \rightarrow \kappa - a_j$ we have

$$V_{\kappa}^{g;p}(n) = \sum_{\substack{0 < a_g < \dots < a_1 < \kappa \\ \mathbf{b} \in [d]^g}} K_{d;a_g p}^{(p)}(\mathbf{b}) = \sum_{\substack{0 < a_g < \dots < a_1 < \kappa \\ \mathbf{s} \in \text{DW}(d, n-1)}} \mathcal{H}_{a_g p}^{(p)}(\mathbf{s}).$$

For each $\mathbf{s} \in \text{DW}(d, n-1)$, let Γ_d be its permutation group (a symmetry group of d letters), $\text{Orb}(\mathbf{s})$ its orbit under Γ_d , and $\text{Stab}(\mathbf{s})$ its stabilizer, i.e., the subgroup of all of the permutations that fix \mathbf{s} . It is well-known from group theory that $|\text{Orb}(\mathbf{s})| \cdot |\text{Stab}(\mathbf{s})| = |\Gamma_d| = d!$. Thus we have

$$\begin{aligned} V_{\kappa}^{g;p}(n) &= \sum_{\substack{0 < a_g < \dots < a_1 < \kappa \\ \mathbf{s} \in \text{DW}(d, n-1)}} \frac{1}{|\text{Orb}(\mathbf{s})|} \sum_{\mathbf{t} \in \text{Orb}(\mathbf{s})} \mathcal{H}_{a_g p}^{(p)}(\mathbf{t}) \\ &= \sum_{\substack{0 < a_g < \dots < a_1 < \kappa \\ \mathbf{s} \in \text{DW}(d, n-1)}} \frac{1}{|\text{Orb}(\mathbf{s})|} \cdot \frac{U_{0;a_g}^{(p)}(\mathbf{s})}{|\text{Stab}(\mathbf{s})|} = \sum_{\substack{0 < a_g < \dots < a_1 < \kappa \\ \mathbf{s} \in \text{DW}(d, n-1)}} \frac{U_{0;a_g}^{(p)}(\mathbf{s})}{d!}. \end{aligned} \quad (11)$$

Since $d = n - g - 1$ and $B_{p-n} = 0$ for even n , by Lemma 2.1,

$$U_{0;a_g}^{(p)}(\mathbf{s}) \equiv a_g(-1)^{g+1}(d-1)!(n-1)\frac{B_{p-n}}{n}p \pmod{p^2}.$$

Noticing that $|\text{DW}(d, n-1)| = \binom{n-2}{d-1}$, we get

$$\begin{aligned} V_{\kappa}^{g;p}(n) &\equiv \sum_{0 < a_g < \dots < a_2 < a_1 < \kappa} a_g(-1)^{g+1} \binom{n-2}{d-1} \frac{n-1}{d} \frac{B_{p-n}}{n} p \pmod{p^2} \\ &\equiv \sum_{0 < a_g < \dots < a_2 < a_1 < \kappa} a_g(-1)^{g+1} \binom{n-1}{g} \frac{B_{p-n}}{n} p \pmod{p^2}. \end{aligned}$$

So the lemma follows from (12) at once. \square

Lemma 5.2. *For any positive integers $i \leq g < \kappa$, we have*

$$\sum_{0 < a_1 < \dots < a_g < \kappa} a_i = i \sum_{a=1}^{\kappa-1} \binom{a}{g}.$$

In particular, if $i = g$ then we have

$$\sum_{0 < a_g < \dots < a_2 < a_1 < \kappa} a_g = \binom{\kappa}{g+1} \quad (12)$$

Proof. Clearly

$$\sum_{0 < a_1 < \dots < a_i} 1 = \binom{a_i - 1}{i - 1} = \frac{i}{a_i} \binom{a_i}{i}$$

is the number of ways to choose $i - 1$ distinct positive integers from $1, 2, \dots, a_i - 1$. The lemma follows quickly from an induction on g by using the well-known identity

$$\sum_{0 < a_i < a_{i+1}} \binom{a_i}{i} = \binom{a_{i+1}}{i+1}.$$

In particular, if $i = g$ then we may take $a_{i+1} = \kappa$ to prove (12). \square

Lemma 5.3. *We have*

$$M_{\kappa}^{g;p}(n) \equiv 0 \pmod{p}.$$

Proof. Again we let $d = n - g - 1$. By the definition and Lemma 5.2,

$$\begin{aligned} M_{\kappa}^{g;p}(n) = & \sum_{\substack{0 < a < \kappa \\ 1 \leq b_1 \leq \dots \leq b_g \leq d}} \binom{a}{g} \sum_{\substack{0 < u_1 < \dots < u_d < (\kappa - a)p \\ u_1, u_2, \dots, u_d \in \mathcal{P}_p}} \frac{1}{u_1 \dots u_d u_{b_1} \dots u_{b_g}} \left(\frac{1}{u_{b_1}} + \frac{1}{u_{b_1+1}} + \right. \\ & \left. \dots + \frac{1}{u_{b_2}} + \frac{2}{u_{b_2}} + \dots + \frac{2}{u_{b_3}} + \frac{3}{u_{b_3}} + \dots + \frac{g-1}{u_{b_g}} + \frac{g}{u_{b_g}} + \dots + \frac{g}{u_d} \right). \end{aligned} \quad (13)$$

Because of the terms in the parenthesis, we see that each $\mathbf{b} \in [d]^g$ may produce more than one p -restricted MHSs of weight n . Hence,

$$M_{\kappa}^{g;p}(n) = \sum_{0 < a < \kappa} \binom{a}{g} \sum_{\mathbf{s} \in \text{DW}(d, n)} m(\mathbf{s}) \mathcal{H}_{(\kappa - a)p}^{(p)}(\mathbf{s}).$$

We now show that the multiplicity $m(\mathbf{s}) = g(g+1)/2$ for all \mathbf{s} . For simplicity, we set

$$\mathbf{1} = (l_1, \dots, l_d) = (s_1 - 1, \dots, s_d - 1).$$

The idea is to subtract 1 from a component $s_j > 1$ of \mathbf{s} and consider the corresponding $\mathbf{b}(j)$ using the 1-1 correspondence ρ defined by (10). Every such $\mathbf{b}(j)$ produced will lead to a p -restricted MHS $\mathcal{H}_{(\kappa - a)p}^{(p)}(\mathbf{s})$ with some multiplicity due to the possible repetition of $1/u_j$ -term in the parenthesis of (13). Suppose $s_j \geq 2$. Then we get the corresponding

$$\mathbf{b}(j) = (b_1, \dots, b_g) = (1_{l_1}, 2_{l_2}, \dots, (j-1)_{l_{j-1}}, j_{l_j-1}, (j+1)_{l_{j+1}}, \dots, d_{l_d}).$$

Set $t = l_1 + \dots + l_{j-1}$. Then we see that $b_{t+i} = j$ for all $i = 1, \dots, l_j - 1$. So the contribution to the multiplicity of $m(\mathbf{s})$, denoted by $m_j(\mathbf{s})$, by this particular $\mathbf{b}(j)$ is given by the coefficient of $1/u_j$ in the above (note that $1/u_j$ repeats l_j times with increasing numerators), namely,

$$m_j(\mathbf{s}) = \mu_j(\mathbf{1}) := t + \sum_{i=1}^{l_j-1} (t+i) = \left(l_1 + \dots + l_{j-1} + \frac{l_j-1}{2} \right) l_j.$$

Remarkably, this is still true even if $s_j = 1$, i.e., $l_j = 0$, because $\mathbf{b}(j)$ doesn't exist in this case while $m_j(\mathbf{s}) = 0$ according to the formula.

We now show that $\mu(\mathbf{1})$ only depends on $|\mathbf{1}| = n - d = g + 1$. Indeed, let $\mathbf{l}' = (l_1 - 1, \dots, l_{i-1}, l_i + 1, l_{i+1}, \dots, l_d)$ for some $i \geq 2$ and let $r_j = \mu_j(\mathbf{1}) - \mu_j(\mathbf{l}')$. If $j = 1$, we have

$$r_1 = \left(\frac{l_1 - 1}{2} \right) l_1 - \left(\frac{l_1 - 2}{2} \right) (l_1 - 1) = l_1 - 1.$$

For $1 < j < i$,

$$r_j = \left(l_1 + l_2 + \cdots + l_{j-1} + \frac{l_j - 1}{2} \right) l_j - \left(l_1 - 1 + l_2 + \cdots + l_{j-1} + \frac{l_j - 1}{2} \right) l_j = l_j.$$

For $j = i$,

$$\begin{aligned} r_i &= \left(l_1 + l_2 + \cdots + l_{i-1} + \frac{l_i - 1}{2} \right) l_i - \left(l_1 - 1 + l_2 + \cdots + l_{i-1} + \frac{l_i}{2} \right) (l_i + 1) \\ &= 1 - (l_1 + \cdots + l_{i-1}). \end{aligned}$$

For $j > i$,

$$r_j = \left(l_1 + l_2 + \cdots + l_{j-1} + \frac{l_j - 1}{2} \right) l_j - \left(l_1 - 1 + l_2 + \cdots + l_{j-1} + 1 + \frac{l_j - 1}{2} \right) l_j = 0.$$

Therefore

$$\mu(\mathbf{1}) - \mu(\mathbf{1}') = \sum_{j=1}^d \left(\mu_j(\mathbf{1}) - \mu_j(\mathbf{1}') \right) = \sum_{j=1}^d r_j = 0.$$

The upshot is that $m(\mathbf{s}) = \sum_{j=1}^d m_j(\mathbf{s}) = \sum_{j=1}^d \mu_j((g+1, 0, \dots, 0)) = \mu_1((g+1, 0, \dots, 0)) = g(g+1)/2$ as desired. Consequently, using the idea to derive (11), we see that

$$\begin{aligned} M_{\kappa}^{g;p}(n) &= \frac{g(g+1)}{2} \sum_{0 < a < \kappa} \binom{a}{g} \sum_{\mathbf{s} \in \text{DW}(d,n)} \mathcal{H}_{(\kappa-a)p}^{(p)}(\mathbf{s}) \\ &= \frac{g(g+1)}{2} \sum_{0 < a < \kappa} \binom{a}{g} \sum_{\mathbf{s} \in \text{DW}(d,n)} \frac{U_{0;\kappa-a}^{(p)}(\mathbf{s})}{d!} \equiv 0 \pmod{p} \end{aligned}$$

by Lemma 2.1. □

Lemma 5.4. *We have*

$$P_{\alpha;\kappa}^{g;p}(n) \equiv P_{0;\kappa}^{g;p}(n) \pmod{p^2}.$$

Proof. As before we let $d = n - g - 1$. Define

$$\begin{aligned} E_{\kappa}^{g;p}(n) &:= \sum_{\substack{0 < a_1 < \cdots < a_g < \kappa \\ 1 \leq b_1 \leq \cdots \leq b_g \leq d}} \sum_{\substack{0 < u_1 < \cdots < u_d < (\kappa - a_g)p \\ u_1, u_2, \dots, u_d \in \mathcal{P}_p}} \frac{1}{u_1 \cdots u_d u_{b_1} \cdots u_{b_g}} \left(\frac{1}{u_1} + \cdots + \frac{1}{u_d} \right), \\ F_{\kappa}^{g;p}(n) &:= \sum_{\substack{0 < a_1 < \cdots < a_g < \kappa \\ 1 \leq b_1 \leq \cdots \leq b_g \leq d}} \sum_{\substack{0 < u_1 < \cdots < u_d < (\kappa - a_g)p \\ u_1, u_2, \dots, u_d \in \mathcal{P}_p}} \frac{1}{u_1 \cdots u_d u_{b_1} \cdots u_{b_g}} \left(\frac{1}{u_{b_1}} + \cdots + \frac{1}{u_{b_g}} \right). \end{aligned}$$

Then it is easy to see that

$$P_{0;\kappa}^{g;p}(n) - P_{\alpha;\kappa}^{\kappa}(n; p) \equiv \alpha p \left(E_{\kappa}^{g;p}(n) + F_{\kappa}^{g;p}(n) \right) \pmod{p^2}. \quad (14)$$

Indeed, in the definition (6) we may replace every u_j by $u_j + \alpha p$. Then by geometric expansion in the p -adic integer ring \mathbb{Z}_p , we see that

$$\frac{1}{u_j + \alpha p} \equiv \frac{1}{u_j} \left(1 - \frac{\alpha p}{u_j} \right), \quad \frac{1}{u_j + (\alpha + a_i)p} \equiv \frac{1}{u_j} \left(1 - \frac{(\alpha + a_i)p}{u_j} \right) \pmod{p^2}, \quad (15)$$

which quickly imply (14).

We first prove that

$$E_{\kappa}^{g;p}(n) \equiv 0 \pmod{p}. \quad (16)$$

By the proof of Lemma 5.1 we see that there is a 1-1 correspondence between $[d]^g$ and $\text{DW}(d, n-1)$, where $[d]^g$ is the set of g -tuples of integers in $\{1, \dots, d\}$ and $\text{DW}(d, n-1) \subset \mathbb{N}^d$ is the set of d -tuples \mathbf{s} with $|\mathbf{s}| = n-1$. Let the height of \mathbf{s} , denoted by $\text{ht}(\mathbf{s})$, be the number of components of \mathbf{s} which are greater than 1. Let $\text{DW}(d, n, h)$ be the subset of height h elements of $\text{DW}(d, n)$. Since $n-d = g+1 \geq 1$ the height of every element in $\text{DW}(d, n)$ is at least 1. Define

$$\begin{aligned} \lambda_j : \text{DW}(d, n-1) &\longrightarrow \text{DW}(d, n) \\ (s_1, \dots, s_d) &\longmapsto (s_1, \dots, s_{j-1}, s_j + 1, s_{j+1}, \dots, s_d). \end{aligned}$$

It is obvious that the union of the images of λ_j , as a multi-set, covers every element of $\text{DW}(d, n, h)$ exactly h times. Note further that the set $\text{DW}(d, n, h)$ is invariant under every permutation of the components of its elements. Using the same idea to derive (11), we get

$$\begin{aligned} E_{\kappa}^{g;p}(n) &= \sum_{0 < a_1 < \dots < a_g < \kappa} \sum_{h=1}^d h \sum_{\mathbf{s} \in \text{DW}(d, n, h)} \mathcal{H}_{(\kappa-a_g)p}^{(p)}(\mathbf{s}) \\ &= \sum_{0 < a_1 < \dots < a_g < \kappa} \sum_{h=1}^d h \sum_{\mathbf{s} \in \text{DW}(d, n, h)} \frac{U_{0; \kappa-a}^{(p)}(\mathbf{s})}{d!} \equiv 0 \pmod{p} \end{aligned}$$

by Lemma 2.1.

We now prove that

$$F_{\kappa}^{g;p}(n) \equiv 0 \pmod{p}. \quad (17)$$

We modify the idea used in the proof of Lemma 5.3. Recall that for any $\mathbf{s} = (s_1, \dots, s_d) \in \text{DW}(d, n-1)$, we set $\rho^{-1}(\mathbf{s}) = (1_{l_1}, 2_{l_2}, \dots, d_{l_d})$ where $l_j = s_j - 1$ for all $j = 1, \dots, d$. So we argue similarly as in the proof of Lemma 5.3 and see that

$$F_{\kappa}^{g;p}(n) = \sum_{0 < a_1 < \dots < a_g < \kappa} \sum_{\mathbf{s} \in \text{DW}(d, n)} m(\mathbf{s}) \mathcal{H}_{(\kappa-a_g)p}^{(p)}(\mathbf{s}),$$

where the multiplicity

$$m(\mathbf{s}) = l_1 + l_2 + \dots + l_d = g$$

which is independent of \mathbf{s} . Thus

$$\begin{aligned} F_{\kappa}^{g;p}(n) &= \sum_{0 < a_1 < \dots < a_g < \kappa} g \sum_{\mathbf{s} \in \text{DW}(d, n)} \mathcal{H}_{(\kappa-a_g)p}^{(p)}(\mathbf{s}) \\ &= \sum_{0 < a_1 < \dots < a_g < \kappa} g \sum_{\mathbf{s} \in \text{DW}(d, n)} \frac{U_{0; \kappa-a}^{(p)}(\mathbf{s})}{d!} \equiv 0 \pmod{p} \end{aligned}$$

by Lemma 2.1.

Finally, the lemma follows from (14), (16) and (17). \square

We are now ready to prove Proposition 3.1. By the definition, we have

$$\begin{aligned}
P_{0;\kappa}^{g;p}(n) &= \sum_{\substack{0 < a_1 < \dots < a_g < \kappa \\ 0 < b_1 \leq \dots \leq b_g < n-g}} \sum_{\substack{0 < u_1 < \dots < u_d < (\kappa - a_g)p \\ u_1, u_2, \dots, u_d \in \mathcal{P}_p}} \frac{1}{u_1 u_2 \dots u_{b_1} (u_{b_1} + a_1 p) (u_{b_1+1} + a_1 p)} \\
&\quad \dots \frac{1}{(u_{b_2} + a_1 p) (u_{b_2} + a_2 p) \dots (u_{b_g} + a_{g-1} p) (u_{b_g} + a_g p) \dots (u_d + a_g p)} \\
&\equiv V_{\kappa}^{g;p}(n) - p M_{\kappa}^{g;p}(n) \pmod{p^2}
\end{aligned}$$

by (15). Thus by Lemma 5.1 and Lemma 5.3

$$P_{0;\kappa}^{g;p}(n) \equiv (-1)^{g+1} \binom{\kappa}{g+1} \binom{n-1}{g} \frac{B_{p-n}}{n} p \pmod{p^2}.$$

So Proposition 3.1 follows from Lemma 5.4.

We can now turn to the proof of the Main Theorem. From Theorem 3.3 and Lemma 2.4 we see that for all $m, n \in \mathbb{N}$, both $R_n^{(m,1)}$ and $S_n^{(m,1)}$ lie in the subalgebra \mathcal{B} of \mathcal{A}_1 generated by \mathcal{A} -Bernoulli numbers. This implies that $S_n^{(m,2)}$ lies in $p\mathcal{B} \subset \mathcal{A}_2$ by Lemma 2.2 (ii), which in turn yields (3) and (2) by Lemma 2.2 (iii) and Lemma 2.3, respectively. We can now conclude the proof of our Main Theorem and the paper.

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